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Quasi-steady flow of a rotating stratified fluid in a sphere

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The steady and quasi-steady motion achieved in a rotating stratified sphere of fluid is studied in the context of a linearized Boussinesq model. In certain parameter ranges an explicit expression is obtained for the flow field as a functional of the surface stress. The non-singular interior solution is used to examine the behaviour of the boundary layer close to the equator. The results agree with previous conclusions about the behaviour of a rotating stratified fluid in simpler geometries. Viewing the problem as a simple model for the interior core of the sun, this work indicates a solar spin-down time that is within the lifetime of the sun.

1. Introduction

Motion in a rotating stratified fluid is of obvious geophysical and astrophysical interest. Theoretical study of flow within a cylinder has been carried out by a number of investigators, including Holton (1965), Pedlosky (1967) and Sakurai (1969), and with particular interest in the solar controversy, Friedlander (1974) and Sakurai, Clark & Clark (1971). In the context of a linearized model they have investigated the mechanisms by which a small perturbation in the boundary conditions are communicated to the interior flow. The purpose of this paper is to examine the flow of a rotating stratified fluid in a sphere; we obtain an explicit expression for the flow field on a long time scale as a functional of the driving stress. The results we obtain agree in overall nature with the general understanding of the problem in cylindrical geometry.

A homogeneous rotating fluid responds to a change $\Delta\Omega$ in the rotation rate in a time of order $E^{-\frac{1}{2}}\Omega^{-1}$, where E is the Ekman number, defined in §2. The parameter $E^{-\frac{1}{2}}$ is known as the dimensionless homogeneous spin-up time scale. However in the case of a stratified rotating fluid the angular velocity of each fluid particle is governed by a combination of physical processes with the emphasis depending on the location of the particle as well as the parameter ranges describing the problem. The definition of a stratified spin-up time is by no means universal: a discussion of the meaning of this term is given by Buzyna & Veronis (1971). As they point out, significant changes in the angular velocity of a particle will take place between the homogeneous spin-up time and the diffusion time. In this paper we consider the stratified spin-up time to be the time scale on which the angular velocity of every particle of the fluid has been modified by the applied boundary condition. In our parameter range we find this time scale is of order

$N^2\sigma E^{-1}$ (for definition see § 2), which lies between $E^{-\frac{1}{2}}$ and the viscous diffusion time scale of E^{-1}.

Our particular concern with this problem is motivated by an interest in the solar circulation; we therefore concentrate attention on the case relevant to the solar problem, where the Prandtl number σ is very small and the Brunt–Väisälä frequency N is large. We consider the possibility that the outer shell of the sun is rotating more slowly than the inner core; we model the ensuing viscous coupling by considering a constant stress applied to the surface of a rotating stratified sphere. As the thermal boundary condition, the temperature is held fixed at the surface. We note that, in the parameter ranges we are considering, the thermal boundary condition is unimportant. The nature of the results is the same if the fixed temperature boundary condition is replaced by an insulated boundary condition. Solutions were obtained with the latter boundary condition in Friedlander (1972); the solutions for the cylindrical problem are exhibited with both an insulated and fixed temperature condition in Friedlander (1974).

We find that the flow is driven by a combination of Ekman-layer suction and thermal diffusion. Examination of the equations of motion yields the $O(1)$ flow as an explicit functional of the stress. The solution is such that the interior velocity is always in balance with surface stress to ensure that the radial velocity induced by the Ekman layer is non-singular. In the final section, we use the non-singular interior solution to examine the boundary layer in detail in the equatorial regions. We find that the Ekman layer thickens to a layer of thickness $O(E^{\frac{2}{5}})$; however the role of this corner layer is essentially a passive matching of the interior flow with the prescribed boundary conditions.

2. Equations of motion

We shall describe the motion in terms of the potential-vorticity equation that is derived from the linearized non-dimensional equations describing a Boussinesq fluid.

The basic equations, in the rotating co-ordinate system, describing a viscous heat-conducting fluid, stratified under a potential Φ, are

$$\mathbf{u}_t + 2\hat{\mathbf{k}}\Omega \times \mathbf{u} + \mathbf{u} . \nabla\mathbf{u} + \rho^{-1}\nabla P + \nabla\Phi = \nu\nabla^2\mathbf{u} + (\beta + \tfrac{4}{3}\nu)\nabla\nabla . \mathbf{u}, \tag{2.1}$$

$$\rho_t + \nabla\rho\mathbf{u} = 0, \tag{2.2}$$

$$s_t + \mathbf{u} . \nabla s = \kappa\nabla^2 s + \frac{\nu}{\rho c_p}[\nabla^2\mathbf{u} . \mathbf{u} + 2\nabla . (\nabla \times \mathbf{u}) \times \mathbf{u} - 2\mathbf{u}\nabla\nabla . \mathbf{u} + \nabla \times \mathbf{u} . \nabla \times \mathbf{u} - \tfrac{2}{3}(\nabla . \mathbf{u})^2] + \frac{\beta}{\rho c_p}(\nabla . u)^2, \tag{2.3}$$

together with an equation of state

$$\rho = f(P, s).$$

Here $\mathbf{u}$ denotes velocity, P pressure, s temperature, ρ density, and Ω is the angular velocity of the sphere. The kinematic and kinematic bulk viscosities are ν and β and κ is the coefficient of thermal diffusivity.

We introduce dimensionless variables by writing

$$\mathbf{r} = a\mathbf{r}^*, \quad t = \Omega^{-1}t^*, \quad \mathbf{u} = \Omega a\mathbf{u}^*, \quad \rho = \bar{\rho}\rho^*,$$
$$s = \bar{s}s^*, \quad P = a^2\Omega^2\bar{\rho}P^*, \quad \Phi = \Omega^2a^2\Phi^*.$$

We then linearize about the steady state by writing

$$\mathbf{u}^* = \epsilon\mathbf{u}, \quad \rho^* = \rho_0(\Phi)+\epsilon\rho, \quad P^* = P_0(\Phi)+\epsilon P, \quad s^* = s_0(\Phi)+\epsilon s,$$

where ϵ is the Rossby number, which is assumed to be small.

We shall now assume that the Boussinesq approximation is valid and that the equation of state is such that $f_P = 0$. We model the self-gravitating system by an externally given potential Φ, where

$$\Phi^* = -\tfrac{1}{2}r^2\sin^2\theta+\gamma r^2.$$

Here $\gamma = g/(a\Omega^2)$, and we assume that γ is sufficiently large that the centrifugal contribution can be neglected.

For a convenient simple model we shall take the basic stratification to be such that $\partial s_0/\partial\Phi^*$ and f_s are constants with $f_s = -\alpha$. With a suitable choice of basic scales of density and temperature we may take

$$\rho_0 \simeq 1, \quad \partial s_0/\partial\Phi^* = 1/\gamma, \quad \text{so that} \quad s_0 = r^2.$$

Thus the linearized equations describing the Boussinesq fluid are, in non-dimensional form,

$$\mathbf{u}_t+2\hat{\mathbf{k}}\times\mathbf{u}+\nabla P-2N^2s\mathbf{r} = E\nabla^2\mathbf{u}, \tag{2.4}$$

$$\nabla.\mathbf{u} = 0, \tag{2.5}$$

$$s_t+2\mathbf{u}.\mathbf{r} = (E/\sigma)\nabla^2 s, \tag{2.6}$$

where the Ekman number $E = \nu/\Omega a^2$, the Prandtl number $\sigma = \nu/\kappa$, and the stratification parameter $N^2 = \alpha\gamma$. We shall investigate the long time-scale behaviour of this set of equations when the perturbation from the initial steady state is caused by the following boundary conditions:

$$\left.\begin{aligned} r\frac{\partial}{\partial r}\frac{u_3}{r} = \tau(\theta), \quad r\frac{\partial}{\partial r}\frac{u_2}{r} = 0 \\ u_1 = 0, \quad s = 0 \end{aligned}\right\} \quad \text{on} \quad r = 1, \tag{2.7}$$

where the velocity $\mathbf{u} = u_1\hat{r}+u_2\hat{\theta}+u_3\hat{\phi}$. The initial condition is $P(\mathbf{r},0) = 0$. We assume that the motion generated by these boundary conditions is axisymmetric, i.e. $\partial/\partial\phi = 0$.

Manipulating (2.4), (2.5) and (2.6) by taking $\nabla\times(2.4)$ and substituting (2.5) and (2.6) we obtain an equation of the form

$$\partial\Pi(\mathbf{u},s)/\partial t = EF(\mathbf{u},s), \tag{2.8}$$

where the functionals Π and F are given by

$$\Pi(\mathbf{u},s) = z\frac{\partial}{\partial z}\frac{s}{z}+\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}\frac{\sin\theta}{\cos\theta}u_3 \tag{2.9}$$

and $$F(\mathbf{u},s) = z\frac{\partial}{\partial z}\frac{1}{z}\nabla^2 s+\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}\frac{\sin\theta}{\cos\theta}\left(\nabla^2-\frac{1}{r^2\sin^2\theta}\right)u_3, \tag{2.10}$$

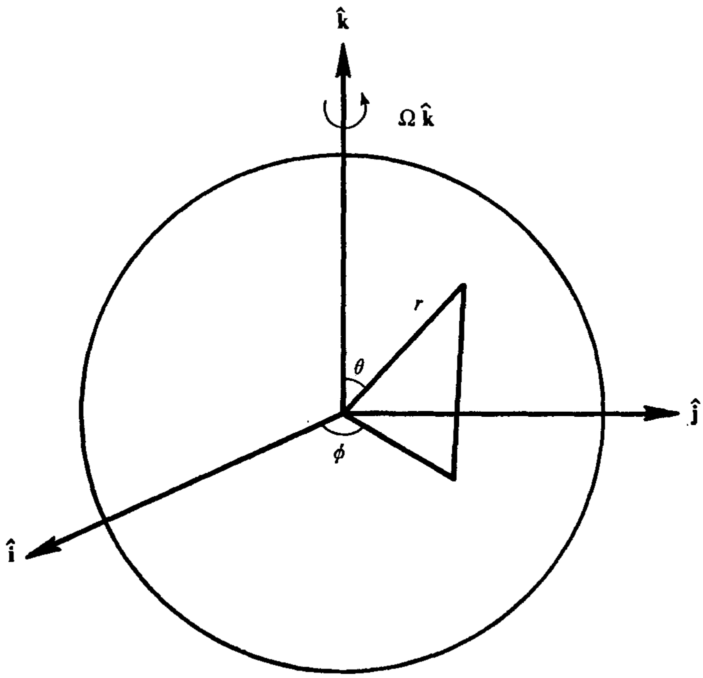

FIGURE 1. $\mathbf{u} = u_1 \hat{\mathbf{r}} + u_2 \hat{\boldsymbol{\theta}} + u_3 \hat{\boldsymbol{\phi}}$.

where $z = r\cos\theta$. (For details of this manipulation see Friedlander 1972.) We note that the functional $\Pi = r\cos\theta \times$ potential vorticity, where the potential vorticity is the function conserved for an inviscid fluid studied by Howard & Siegmann (1969).

3. Stress-driven motion

We shall now examine the problem given by (2.8),

$$\partial\Pi(u_3, s)/\partial t = EF(u_3, s),$$

with boundary conditions representing a steady surface stress $\tau(\theta)\hat{\phi}$ given by (2.7).

Assuming that the viscous coefficient E is very small, and that the diffusive terms can be neglected to first order in the interior, the component equations describing the interior flow are

$$u_{1t} - 2u_3\sin\theta + \partial P/\partial r - 2N^2 sr = 0, \tag{3.1}$$

$$u_{2t} - 2u_3\cos\theta + \frac{1}{r}\frac{\partial P}{\partial\theta} = 0, \tag{3.2}$$

$$u_{3t} + 2u_2\cos\theta + 2u_1\sin\theta = 0, \tag{3.3}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2 u_1 + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta u_2 = 0, \tag{3.4}$$

$$s_t + 2u_1 r = 0. \tag{3.5}$$

The structure of the boundary layer at the surface of the sphere $r = 1$ is essentially the same as the structure of the homogeneous Ekman boundary layer. This fact is noted by Barcilon & Pedlosky (1967), who remark that whenever an Ekman layer is present on a horizontal surface the structure is independent of the stratification. In our problem the spherical boundary is entirely horizontal, being perpendicular to the radial gravitational force. There exists an Ekman layer of thickness $O(E^{\frac{1}{2}})$ characterized by a balance between the Coriolis force and the viscous force. We use the Ekman-layer flow to match the inviscid interior with the stress boundary condition: this procedure gives the familiar Ekman-layer suction condition

$$u_1(1,\theta) = \frac{E}{2\sin\theta}\frac{\partial}{\partial\theta}\frac{\sin\theta}{\cos\theta}\left[\tau(\theta) - r\frac{\partial}{\partial r}\frac{u_3}{r}\right]_{r=1}. \tag{3.6}$$

In the boundary layer the temperature field is $O(E^{\frac{1}{2}})$, thus the thermal boundary condition on the interior flow is, to $O(1)$,

$$s(1,\theta) = 0. \tag{3.7}$$

The Ekman-layer suction condition (3.6) shows that in the interior u_1 is $O(E)$. Hence the balance of terms in the divergence equation (3.4) requires that u_2 is also $O(E)$. Thus in the interior we have to first order in powers of $E^{\frac{1}{2}}$

$$u_1 = O(E), \quad u_2 = O(E), \quad u_3 = O(1), \quad s = O(1), \quad P = O(1).$$

Equations (3.1) and (3.2) then determine the relation between u_3, s and P, namely

$$u_3 = \frac{1}{2r\cos\theta}\frac{\partial P}{\partial\theta} \tag{3.8}$$

and

$$s = \frac{1}{2N^2 z}\frac{\partial P}{\partial z}, \tag{3.9}$$

where $z = r\cos\theta$. These relations essentially say that to $O(1)$ the flow is in geostrophic and hydrostatic balance.

We note that (3.3) and (3.5) require u_3 and s to be independent of time on an $O(1)$ time scale. This is acceptable since we are interested in the long time behaviour, or the quasi-steady flow. Our approach will not describe internal waves. We do not expect these to be present in the problem we are considering, where the initial condition is $P(\mathbf{r}, 0) = 0$, and the driving mechanism is a boundary stress. Substituting (3.8) and (3.9) in the potential-vorticity equation (2.8) gives the equation for the $O(1)$ interior pressure field

$$\partial\Pi(\mathbf{u}(P), s(P))/\partial t = EF(\mathbf{u}(P), s(P)), \tag{3.10}$$

where Π is a second-order spatial differential operator and F is a fourth-order differential operator on the pressure field P.

Returning to the Ekman-layer boundary condition (3.6) we observe from the heat equation that $u_1(1,\theta)$ can be written in terms of s. Since u_1 is $O(E)$ we must consistently include the term representing thermal diffusion in (3.5), thus

$$u_1(1,\theta) = \frac{E}{2\sigma}\nabla^2 s - \frac{1}{2}\frac{\partial s}{\partial t}\bigg|_{r=1}. \tag{3.11}$$

The boundary condition (3.6) then becomes

$$\frac{\partial s}{\partial t} = \frac{E}{\sigma}\nabla^2 s + \frac{E}{\sin\theta}\frac{\partial}{\partial\theta}\frac{\sin\theta}{\cos\theta}\left[r\frac{\partial}{\partial r}\frac{u_3}{r} - \tau(\theta)\right], \tag{3.12}$$

together with the thermal condition $s = 0$ at $r = 1$. Using (3.8) and (3.9) the boundary conditions can be written in terms of the pressure field P.

When we substitute the expressions (2.9) and (2.10) for the functionals Π and F we obtain the problem for the flow on a long time scale in terms of the $O(1)$ interior pressure field. The equation for P is a fourth-order partial differential equation in space which admits the two boundary conditions given by Ekman-layer driving and zero perturbation in the temperature field:

$$\frac{\partial}{\partial t}\left[\frac{z}{N^2}\frac{\partial}{\partial z}\frac{1}{z^2}\frac{\partial P}{\partial z} + \frac{\cos\theta}{r\sin\theta}\frac{\partial}{\partial\theta}\frac{\sin\theta}{\cos^2\theta}\frac{\partial P}{\partial\theta}\right]$$
$$= \frac{E}{N^2\sigma}z\frac{\partial}{\partial z}\frac{1}{z}\nabla^2\frac{1}{z}\frac{\partial P}{\partial z} + E\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}\frac{\sin\theta}{\cos\theta}\left(\nabla^2 - \frac{1}{r^2\sin^2\theta}\right)\left(\frac{1}{r\cos\theta}\frac{\partial P}{\partial\theta}\right), \tag{3.13}$$

with boundary conditions

$$\frac{1}{N^2}\frac{\partial}{\partial t}\frac{1}{z}\frac{\partial P}{\partial z} = \frac{E}{N^2\sigma}\nabla^2\frac{1}{z}\frac{\partial P}{\partial z} + \frac{E}{\sin\theta}\frac{\partial}{\partial\theta}\frac{\sin\theta}{\cos\theta}\left[r\frac{\partial}{\partial r}\frac{1}{r^2\cos\theta}\frac{\partial P}{\partial\theta} - 2\tau\right] \tag{3.14}$$

and

$$z^{-1}\partial P/\partial z = 0 \tag{3.15}$$

at $r = 1$.

As we have remarked, we are particularly interested in the case relevant to the solar problem, where the dimensionless parameter $N^2\sigma$ is very small. (The approximate value appropriate to the sun is $N^2\sigma \sim 10^{-4}$.) We shall consider the parameter ranges

$$E^{\frac{1}{2}} \ll N^2\sigma \ll 1.$$

In this case we note that on a time scale shorter than $O(N^2\sigma E^{-1})$ the problem for P becomes

$$\partial\Pi(\mathbf{u}(P), s(P))/\partial t = 0,$$

with $z^{-1}\partial P/\partial z = 0$ at $r = 1$.

Now Howard & Siegmann (1969) show that the problem

$$\partial\Pi(u, s)/\partial t = 0$$

with s prescribed on the boundary has a unique solution. They show that the geostrophic flow which is the solution to this problem is uniquely determined as the projection of the initial flow onto the space of all geostrophic flows. Thus if the fluid is initially in rigid-body rotation, i.e. the perturbation field is zero for $t \leqslant 0$, the problem has the unique solution $P = 0$. Thus in the problem we are considering there is no quasi-steady geostrophic flow on a time scale shorter than $O(N^2\sigma E^{-1})$. We interpret this to mean that on a shorter time scale the flow is transient and the driving stress has not yet been communicated fully to the interior of the fluid. In our usage of the term, the fluid is not spun-up until the time scale is $O(N^2\sigma E^{-1})$.

We examine the problem on this time scale by writing

$$t = TN^2\sigma E^{-1}.$$

The equation and boundary conditions become

$$\frac{\partial}{\partial T}\left[\frac{z}{N^2}\frac{\partial}{\partial z}\frac{1}{z^2}\frac{\partial P}{\partial z}+\frac{\cos\theta}{r\sin\theta}\frac{\partial}{\partial\theta}\frac{\sin\theta}{\cos^2\theta}\frac{\partial P}{\partial\theta}\right]$$
$$= z\frac{\partial}{\partial z}\frac{1}{z}\nabla^2\frac{1}{z}\frac{\partial P}{\partial z}+N^2\sigma\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}\frac{\sin\theta}{\cos\theta}\left(\nabla^2-\frac{1}{r^2\sin^2\theta}\right)\left(\frac{1}{r\cos\theta}\frac{\partial P}{\partial\theta}\right), \quad (3.16)$$

with boundary conditions

$$\left.\begin{aligned}0 &= \nabla^2\frac{1}{z}\frac{\partial P}{\partial z}+\frac{N^2\sigma}{\sin\theta}\frac{\partial}{\partial\theta}\frac{\sin\theta}{\cos\theta}\left(r\frac{\partial}{\partial r}\frac{P}{r^2\cos\theta}-2\tau(\theta)\right)\\ 0 &= z^{-1}\partial P/\partial z.\end{aligned}\right\}\quad \text{at} \quad r=1. \quad (3.17)$$

Since we are interested in solutions that have developed with time, we shall not consider the full initial-value problem treated by means of Laplace transforms. We shall rather assume a particular form for the time dependence of the solution suggested by the existence of a steady stress driving the motion. We shall assume that the flow can be resolved into a part that grows linearly with T, a steady flow, and a decaying flow represented by a sum of exponential modes. In justification of this assumption we note that in Friedlander (1974) the cylindrical problem was solved using this technique. The results so obtained agreed very well with the flow predicted by a time-dependent numerical model.

We shall therefore write

$$P = TP^0(r,\theta)+P^1(r,\theta)+\Sigma\,\mathscr{P}\rho^{-\lambda T}.$$

Substituting this expression for P into (3.16) and boundary conditions (3.17), and equating coefficients in T gives the problems for P^0, P^1 and $\mathscr{P}$.

Hence the problem for P^0 becomes

$$z\frac{\partial}{\partial z}\frac{1}{z}\nabla^2\frac{1}{z}\frac{\partial P^0}{\partial z}+N^2\sigma\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}\left(\nabla^2-\frac{1}{r^2\sin^2\theta}\right)\left(\frac{1}{r\cos\theta}\frac{\partial P^0}{\partial\theta}\right)=0, \quad (3.18)$$

with
$$\left.\begin{aligned}&\nabla^2\frac{1}{z}\frac{\partial P^0}{\partial z}+\frac{N^2\sigma}{\sin\theta}\frac{\partial}{\partial\theta}\frac{\sin\theta}{\cos\theta}\frac{\partial}{\partial\theta}\frac{1}{r^2\cos\theta}\frac{\partial P^0}{\partial\theta}=0\\ &z^{-1}\partial P^0/\partial z = 0.\end{aligned}\right\}\quad \text{at} \quad r=1. \quad (3.19)$$
and

The solution to this problem is

$$P^0 = \omega r^2\sin^2\theta, \quad \text{giving} \quad u_3 = \omega r\sin\theta.$$

The solution of rigid rotation growing linearly with time is physically plausible. The magnitude of the angular velocity ω will be determined in §4 essentially as the eigenvalue in a singular boundary-value problem. It will be shown that the necessary and sufficient conditions to determine ω and certain constants of integration are the conditions required to prevent singularities in the velocity and vorticity fields. We shall illustrate for stress of the form $\tau(\theta) = A\sin^n\theta$

that the value of ω thus determined is exactly that value which is found by balancing the rate of change of angular momentum of the fluid with the torque due to the applied surface stress.

We note that the above solution for P^0 as a function independent of the z co-ordinate implies that there is no growing temperature variation. It is to be expected that there is no secular growth of mean temperature since the boundary temperature is fixed and viscous dissipation is neglected as a heat source. (The same conclusion would hold in the case of an insulated boundary condition.)

The problem for P^1 is the forced problem given by

$$z\frac{\partial}{\partial z}\frac{1}{z}\nabla^2\frac{1}{z}\frac{\partial P^1}{\partial z}+N^2\sigma\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}\frac{\sin\theta}{\cos\theta}\left(\nabla^2-\frac{1}{r^2\sin\theta}\right)\left(\frac{1}{r\cos\theta}\frac{\partial P^1}{\partial\theta}\right)$$
$$=2\omega r\left(\cos\theta+\frac{1}{\cos\theta}\right), \tag{3.20}$$

with boundary conditions

$$\left.\begin{aligned}\nabla^2\frac{1}{z}\frac{\partial P^1}{\partial z}+\frac{N^2\sigma}{\sin\theta}\frac{\partial}{\partial\theta}\frac{\sin\theta}{\cos\theta}\left[\frac{\partial}{\partial r}\frac{1}{r^2\cos\theta}\frac{\partial P^1}{\partial\theta}-2\tau\right]=0\\ z^{-1}\partial P^1/\partial z=0\end{aligned}\right\}\quad\text{at}\quad r=1. \tag{3.21}$$

We shall now obtain an explicit solution for P^1 as a functional of $\tau(\theta)$ in the case of particular interest, when $N^2\sigma$ is small.

Let us consider an expansion for P^1 in powers of the small parameter $N^2\sigma$ by writing

$$P^1=f(r,\theta)+N^2\sigma g(r,\theta)+\dots.$$

We express P^0 in powers of $N^2\sigma$ by writing

$$\omega=\omega_0+N^2\sigma\omega+\dots.$$

Substituting in (3.20) and equating powers of $N^2\sigma$ gives

$$z\frac{\partial}{\partial z}\frac{1}{z}\nabla^2\frac{1}{z}\frac{\partial f}{\partial z}=2\omega_0 r\left(\cos\theta+\frac{1}{\cos\theta}\right), \tag{3.22}$$

with

$$\nabla^2\frac{1}{z}\frac{\partial f}{\partial z}=0,\quad \frac{1}{z}\frac{\partial f}{\partial z}=0\quad\text{at}\quad r=1. \tag{3.23}$$

Integrating (3.22) with respect to z gives

$$\nabla^2(z^{-1}\partial f/\partial z)=4\omega_0 z^2-2\omega_0 R^2. \tag{3.24}$$

(Symmetry requires that the constant of integration is zero.) The boundary condition (3.23) can be satisfied only if ω_0 is zero; i.e. the angular velocity ω of rigid rotation that grows linearly with T is $O(N^2\sigma)$ with respect to our scaling. Hence the solution to (3.22) with boundary conditions (3.23) is

$$f(r,\theta)=F(R),\quad\text{where}\quad R=r\sin\theta$$

($f(r,\theta)$ is independent of the cylindrical co-ordinate $z=r\cos\theta$). Hence when $N^2\sigma\ll 1$ a form of the Taylor–Proudman theorem is valid to $O(1)$.

Equating terms $O(N^2\sigma)$ gives the problem for $g(r,\theta)$:

$$z\frac{\partial}{\partial z}\frac{1}{z}\nabla^2\frac{1}{z}\frac{\partial g}{\partial z} = -\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}\frac{\sin\theta}{\cos\theta}\left(\nabla^2-\frac{1}{r^2\sin^2\theta}\right)\left(\frac{1}{r\cos\theta}\frac{\partial f}{\partial\theta}\right)+2\omega\left(\frac{R^2+2z^2}{z}\right), \tag{3.25}$$

with boundary conditions

$$\left.\begin{aligned}\nabla^2\frac{1}{z}\frac{\partial g}{\partial z} &= \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\frac{\sin\theta}{\cos\theta}\left[2\tau-\frac{\partial}{\partial r}\frac{1}{r^2\cos\theta}\frac{\partial f}{\partial\theta}\right]\\ z^{-1}\partial g/\partial z &= 0\end{aligned}\right\}\quad \text{at}\quad r=1. \tag{3.26}$$

We express the problem in cylindrical co-ordinates by writing

$$\frac{\partial}{\partial\theta} = z\frac{\partial}{\partial R}-R\frac{\partial}{\partial z},\quad \frac{\partial}{\partial r} = \sin\theta\frac{\partial}{\partial R}+\cos\theta\frac{\partial}{\partial z},$$

where $z = r\cos\theta$, $R = r\sin\theta$. We note that since $f(r,\theta) = F(R)$ we have

$$\frac{1}{r\cos\theta}\frac{\partial f}{\partial\theta} = \frac{\partial F}{\partial R},$$

which is independent of z.

Integrating (3.25) with respect to z gives

$$\nabla^2\frac{1}{z}\frac{\partial g}{\partial z} = -\frac{z^2}{R}\frac{\partial}{\partial R}R\left(\nabla^2-\frac{1}{R^2}\right)\frac{\partial F}{\partial R}+R\left(\nabla^2-\frac{1}{R^2}\right)\frac{\partial F}{\partial R}-2\omega R^2+4\omega z^2+H(R)z, \tag{3.27}$$

where $H(R)$ is the function of integration. We shall now evaluate (3.27) at $r = 1$ by writing $z^2 = 1-R^2$. This gives the following expression for $\nabla^2(z^{-1}\partial g/\partial z)$ at $r = 1$:

$$\left.\nabla^2\frac{1}{z}\frac{\partial g}{\partial z}\right|_{r=1} = -\frac{(1-R^2)}{R}\frac{\partial}{\partial R}R\left(\nabla^2-\frac{1}{R^2}\right)\frac{\partial F}{\partial R}+R\left(\nabla^2-\frac{1}{R^2}\right)\frac{\partial F}{\partial R} +4\omega-6\omega R^2+H(R)\left(\pm(1-R^2)^{\frac{1}{2}}\right). \tag{3.28}$$

Assuming the stress τ is symmetric in the northern and southern hemispheres, we can write the surface stress as a function of R. Hence the boundary condition (3.26) gives us the expression

$$\left.\nabla^2\frac{1}{z}\frac{\partial g}{\partial z}\right|_{r=1} = \frac{2\tau(R)}{R(1-R^2)}+\frac{2\partial\tau}{\partial R}-\frac{\partial}{\partial R}R^2\frac{\partial}{\partial R}\frac{1}{R}\frac{\partial F}{\partial R}-\frac{R}{(1-R^2)}\frac{\partial}{\partial R}\frac{1}{R}\frac{\partial F}{\partial R}. \tag{3.29}$$

Thus we have two expressions for $[\nabla^2(z^{-1}\partial g/\partial z)]_{r=1}$ that are functions of the cylindrical radius R only. Equating these expressions (3.28) and (3.29) we see that symmetry requires the function of integration $H(R)$ to be zero. We have thus obtained an ordinary differential equation for $F(R)$ that holds on the boundary of the sphere $r = 1$. Since the equation involves only the co-ordinate R it must hold throughout the sphere; thus we have reduced the first-order problem to that of solving an ordinary differential equation for $F(R)$, namely

$$-\frac{(1-R^2)}{R}\frac{\partial}{\partial R}R\left(\nabla^2-\frac{1}{R^2}\right)\frac{\partial F}{\partial R}+R\left(\nabla^2-\frac{1}{R^2}\right)\frac{\partial F}{\partial R}+\frac{\partial}{\partial R}R^2\frac{\partial}{\partial R}\frac{1}{R}\frac{\partial F}{\partial R}+\frac{R}{(1-R^2)}\frac{\partial}{\partial R}\frac{1}{R}\frac{\partial F}{\partial R} = -4\omega+6\omega R^2+\frac{2\tau}{R(1-R^2)}+2\frac{\partial\tau}{\partial R}. \tag{3.30}$$

Manipulation of the equation enables us to express $F(R)$ as an integral functional of the stress $\tau(R)$:

$$F(R) = -\int R\,dR \int \frac{dR}{R^3(1-R^2)^{\frac{1}{2}}} \int R\,dR \int \frac{d}{dR}\left[\frac{R\tau}{(1-R^2)^{\frac{1}{2}}} - \omega R(1-R^2)^{\frac{1}{2}}\right] dR. \quad (3.31)$$

Thus for any symmetric stress $\tau(R)$ we can determine the $O(1)$ steady component of the pressure field from (3.31). The velocity and temperature fields are then computed from (3.8) and (3.9).

Once the function $F(R)$ is known, $g(r, \theta)$ can be determined by solving (3.27) for $\nabla^2(z^{-1}\,\partial g/\partial z)$ with the thermal boundary condition

$$z^{-1}\partial g/\partial z = 0 \quad \text{at} \quad r = 1.$$

Hence

$$\frac{1}{z}\frac{\partial g}{\partial z} = R\frac{\partial F}{\partial R} - \frac{z^2}{R}\frac{\partial}{\partial R}R\frac{\partial F}{\partial R} + \omega R^2\left(z^2 - \frac{R^2}{4}\right) + X(r, \theta), \quad (3.32)$$

where $X(r, \theta)$ is a harmonic function chosen to satisfy the boundary condition

$$\frac{\partial}{\partial r}\left(\frac{1}{z}\frac{\partial g}{\partial z}\right) = 0 \quad \text{at} \quad r = 1. \quad (3.33)$$

We remark that the $O(1)$ solution for the pressure field $F(R)$ is independent of z, thus from (3.9) the $O(1)$ perturbation temperature is zero. The fact that the flow to $O(1)$ satisfies the Taylor–Proudman theorem is to be expected since we have made use of the assumption that $N^2\sigma$ is small. For a weakly stratified fluid (i.e. $N^2 \ll 1$) it is reasonable to obtain no $O(1)$ perturbation in the temperature. We find that the temperature field s is $O(N^2\sigma)$ and given by (3.32).

The long term solution that we have obtained is given by

$$P = N^2\sigma\omega r^2 \sin^2\theta T + F(R) + N^2\sigma g(r, \theta) + O(E^{\frac{1}{2}}),$$

where (3.31) gives $F(R)$ as an integral operator on the surface stress, and

$$TN^2\sigma E^{-1} = t.$$

Thus, in terms of the unscaled dimensionless time variable t, we have

$$P = tE\omega r^2 \sin^2\theta + F(R) + N^2\sigma g(r, \theta) + O(E^{\frac{1}{2}}). \quad (3.34)$$

Hence on a time scale shorter than the viscous time (i.e. $t < O(E^{-1})$), the transient term that represents secular growth is of smaller order than $F(R)$. Therefore, on a time scale where $O(N^2\sigma E^{-1}) < t < O(E^{-1})$, the flow to $O(1)$ is steady and $P = F(R)$. Perhaps it would be more correct to call this solution quasi-steady since it is slowly modified by the time-dependent term. On the long time scale when t is greater than $O(E^{-1})$, this time-dependent term becomes dominant and

$$P = Et\omega r^2 \sin^2\theta, \quad \text{where} \quad Et > O(1).$$

Our model predicts that the eventual state of the flow is rigid rotation with linearly growing angular velocity. However, when the angular velocity becomes sufficiently large (i.e. of the order of the inverse of the Rossby number), the initial linear approximation will no longer be valid. We therefore have an upper and a lower bound on the time scale for which (3.34) represents the asymptotic solution to stress-driven flow of a stratified fluid in a rotating sphere.

4. Solution for particular values of the stress

In the previous section we obtained the $O(1)$ solution for the pressure field in terms of the applied stress $\tau(\theta)$. We shall now illustrate for particular symmetric values of $\tau(\theta)$ how the integrals are evaluated and the constants of integration determined. We shall show that the conditions that the velocity and vorticity fields be non-singular at the poles and the equator are the necessary and sufficient conditions to determine the constants of integration and the angular velocity ω. This value of ω is exactly the value required by the balance between the rate of change of angular momentum and the torque due to the surface stress $\tau(\theta)$.

The steady $O(1)$ component of the pressure field is given by (3.31), hence the velocity field u_3 is given by

$$u_3 = -R\int\frac{dR}{R^3(1-R^2)^{\frac{1}{2}}}\int R\,dR\int\frac{d}{dR}\left[\frac{R\tau}{(1-R^2)^{\frac{1}{2}}}-\omega R^2(1-R^2)^{\frac{1}{2}}\right]dR. \qquad (4.1)$$

We shall now indicate how the magnitude of ω and the constants of integration can be determined. When the stress has the form

$$\tau = A\sin^n\theta \quad \text{or} \quad \tau = A\sin^n\theta\,|\cos\theta|$$

the above constants are exactly determined from the condition of no singularities in the velocity and vorticity fields at the pole and at the equator.

Let us first consider

$$\tau = AR^n \quad \text{at} \quad r = 1.$$

To investigate the integral (4.1) we introduce a change of co-ordinates by writing $R = \sin x$. The expression (4.1) for u_3 then becomes

$$u_3 = -\sin x\int\frac{dx}{\sin^3 x}\int(A\sin^{n+2}x-\omega\sin^3x\cos^2x)\,dx$$
$$+(c_0+c_1)\sin x\log\left|\frac{1+\cos x}{\sin x}\right|+\frac{c_1\cos x}{\sin x}+c\sin x, \qquad (4.2)$$

where c, c_0 and c_1 are constants of integration.

Case (a): *n is odd.* Here the integral

$$\int[A\sin^{n+2}x-\omega\sin^3x(1-\sin^2x)]\,dx$$

can be written in terms of $\int\sin^3x\,dx$ as

$$A\left[-\cos x\sum_{k=1}^{\frac{1}{2}(n-1)}\frac{(n+2)!}{(n-2k+3)!}\prod_{j=0}^{k-1}(n+2-2j)^{-2}\sin^{(n-2k+3)}x\right]$$
$$+\left[\frac{A(n+2)!}{3!}\prod_{j=0}^{\frac{1}{2}(n-3)}(n+2-2j)^{-2}-\frac{\omega}{5}\right]\int\sin^3x\,dx-\frac{\omega}{5}\sin^4x\cos x.$$

Thus from (4.2)

$$u_3 = A\left[\sum_{k=1}^{\frac{1}{2}(n-1)} \frac{(n+2)!}{(n-2k+3)!}\prod_{j=0}^{k-1}(n+2-2j)^{-2}\frac{\sin^{n-2k-2}x}{n-2k+1}\right]$$
$$-\left[\frac{A(n+2)!}{3!}\prod_{j=0}^{\frac{1}{2}(n-3)}(n+2-2j)^{-2}-\frac{\omega}{5}\right]\sin x\int\frac{dx}{\sin^3 x}\int\sin^3 x\,dx+\frac{\omega}{10}\sin^3 x$$
$$+(c_0+c_1)\sin x\log\left|\frac{1+\cos x}{\sin x}\right|+\frac{c_1\cos x}{\sin x}+c\sin x. \tag{4.3}$$

Hence the velocity and vorticity fields are non-singular if we choose

$$c_1 = c_0 = 0,$$
$$\omega = 5A\frac{(n+2)!}{3!}\prod_{j=0}^{\frac{1}{2}(n-3)}(n+2-2j)^{-2}.$$

Now the angular velocity ω is determined by the balance between the rate of change of angular momentum and the torque due to the surface stress. Thus

$$\frac{d}{dT}\left[T\int_0^{2\pi}\int_0^{\pi}\int_0^1 \omega r^4\sin^3\theta\,dr\,d\theta\,d\phi\right] = \int_0^{2\pi}\int_0^{\pi}\sin^2\theta\tau\,d\theta\,d\phi.$$

Writing $\tau = A\sin^n\theta$ we obtain

$$\frac{\omega}{5}\int_0^{\pi}\sin^3\theta\,d\theta = A\frac{(n+2)!}{3!}\prod_{j=0}^{\frac{1}{2}(n-3)}(n+2-2j)^{-2}\int_0^{\pi}\sin^3\theta\,d\theta.$$

Hence

$$\omega = 5A\frac{(n+2)!}{3!}\prod_{j=0}^{\frac{1}{2}(n-3)}(n+2-2j)^{-2}.$$

Thus the value of ω prescribed by the angular momentum balance is exactly that value which ensures the flow is non-singular within the sphere.

For example when $\tau = A\sin\theta$ we have $\omega = 5A$ and the $O(1)$ velocity field is

$$u_3 = Et5Ar\sin\theta+\tfrac{1}{2}Ar^3\sin^3\theta+cr\sin\theta, \tag{4.4}$$

where c depends on the initial conditions.

Case (b): n is even. Here the integral

$$\int A\sin^{n+2}x\,dx$$

can be expressed in terms of $\int\sin^2 x\,dx$. This leads to a slightly more complicated balance between ω and A in order to remove possible singularities in u_3 and $R^{-1}\partial(Ru_3)/\partial R$ at $R = 0$ and $R = 1$. Following an analysis similar to case (a) we obtain

$$\omega = (n+2)!\prod_{j=0}^{\frac{1}{2}n}(n+2-2j)^{-2}\frac{15\pi}{8}A,$$

with $c_0 = 0$ and $c_1 = -\frac{1}{15}\omega$. Once again this is the value of the angular velocity ω prescribed by the balance between the applied torque and the rate of increase of angular momentum.

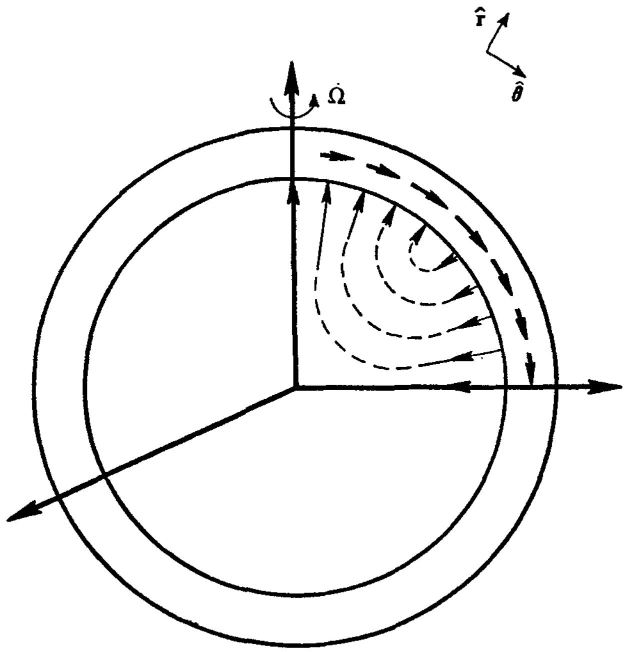

FIGURE 2. Sketch of circulation driven by the stress $\tau(\theta) = A\sin\theta$.

In the case of a symmetric stress of the form $A\sin^n\theta\,|\cos\theta|$ a similar process determines the velocity field. Again this non-singular flow is consistent with the angular momentum fed into the fluid by the surface stress.

We are now in the position to sketch the flow pattern within the sphere: we shall do this for a stress $\tau(\theta) = A\sin\theta$.

From the classical Ekman-layer analysis the $\hat{u}_2$ component of velocity is given by

$$\hat{u}_2 = \pm\left(\left[\tau - r\frac{\partial}{\partial r}\frac{u_3}{r}\right]_{r=1}\Big/ 2\,|\cos\theta|^{\frac{1}{2}}\right) \quad \text{at} \quad \eta = 0.$$

We have calculated in this case that the interior velocity is

$$u_3 = Et5Ar\sin\theta + Cr\sin\theta + \tfrac{1}{2}Ar^3\sin^3\theta.$$

Thus
$$\hat{u}_2(0,\theta) = \pm\tfrac{1}{2}A\sin\theta\,|\cos\theta|^{\frac{3}{2}}. \tag{4.5}$$

Also the interior radial component of velocity is given by

$$u_1 = \frac{1}{2\sin\theta}\frac{\partial}{\partial\theta}\frac{\sin\theta}{\cos\theta}\left[\tau - r\frac{\partial}{\partial r}\frac{u_3}{r}\right]_{r=1}$$
$$= \tfrac{1}{2}A[2\cos^2\theta - \sin^2\theta]. \tag{4.6}$$

Thus figure 2 is a sketch of the flow pattern.

We remark that the Ekman-layer mechanism causes interior upwelling at the poles (for positive stress) which is compensated by interior downwelling at the equator. Within the boundary layer there is no flux of fluid across the equator or at the pole. The total flux of fluid from the Ekman layer into the interior is zero in each quadrant. Thus the fluid moves in a cellular pattern: it enters the Ekman layer at polar latitudes and returns with increased angular momentum at equatorial latitudes.

5. The equatorial Ekman layer

In the previous section, we determined the quasi-steady flow to first order as a functional of the applied surface stress $\tau(\theta)$. This result was obtained assuming the velocity boundary conditions were satisfied via an Ekman-type boundary layer. As noted by Barcilon & Pedlosky (1967), the structure of the Ekman layer is not modified by stratification to first order. Hence, in the region away from the equator, the stream-function equation describing the boundary layer is given by the classical Ekman-layer balance between viscous and Coriolis forces. The equation is

$$-4\cos^2\theta\,\partial^2\psi/\partial\eta^2 = \partial^6\psi/\partial\eta^6, \tag{5.1}$$

where the stream function ψ is defined by

$$u_1 = \frac{-1}{\sin\theta}\frac{\partial\psi}{\partial\theta}, \quad u_2 = \frac{-1}{\sin\theta}\frac{\partial\psi}{\partial\eta}, \tag{5.2}$$

and the boundary-layer co-ordinate is given by

$$E^{\frac{1}{2}}\eta = 1-r.$$

From (5.1), we observe that the thickness of the Ekman layer is

$$O(E^{\frac{1}{2}}\,|\cos\theta|^{-\frac{1}{2}}).$$

Thus, the boundary layer blows up close to the equator, and we expect the Ekman-layer balance to break down.

Now the interior solution obtained in §4 is driven by Ekman-layer suction and hence is not strictly valid in the region close to the equator. Since the interior solution has been constructed to be non-singular at the equator, we extend this solution to the equator. We shall show that the equatorial boundary layer admits the matching of the interior velocity with the imposed stress boundary condition. Thus, in the context we are considering, the role of the equatorial layer is essentially a passive matching of the interior flow, driven by the Ekman layer, with the surface boundary conditions. In this manner, we shall have justified the extension of the interior flow to the equator.

We examine the boundary layer in the equatorial region by writing

$$E^b\zeta = \tfrac{1}{2}\pi - \theta,$$

giving $\quad \cos\theta \simeq E^b\zeta, \quad \sin\theta \simeq 1 \quad \text{and} \quad E^a\eta = 1-r.$

In terms of ζ and η, the boundary-layer equation becomes

$$-4\left[-E^{-b}\frac{\partial}{\partial\zeta} + E^{b-a}\zeta\frac{\partial}{\partial\eta}\right]^2\psi - 4N^2\sigma E^{-2b}\frac{\partial^2\psi}{\partial\zeta^2} = E^2\left[E^{-2a}\frac{\partial^2}{\partial\eta^2} + E^{-2b}\frac{\partial^2}{\partial\zeta^2}\right]^3\psi. \tag{5.3}$$

Examination of (5.3) shows that there is a change in the balance of forces at an interface region where

$$a = \tfrac{2}{5}, \quad b = \tfrac{1}{5}.$$

This corresponds to the results of Stewartson (1966), who examined the equatorial behaviour of a homogeneous fluid between rotating concentric spheres. Hence, at a distance $O(E^{\frac{1}{5}})$ from the equator, the boundary layer thickens to a layer of thickness $O(E^{\frac{2}{5}})$, and the stream-function equation is given by

$$-4\left[-\frac{\partial}{\partial\zeta}+\zeta\frac{\partial}{\partial\eta}\right]^2\psi-4N^2\sigma\frac{\partial^2\psi}{\partial\zeta^2}=\frac{\partial^6\psi}{\partial\eta^6}. \tag{5.4}$$

We shall find an approximate solution to (5.4) with suitable boundary conditions. The approach we shall take is the following. We note that when ζ is large, i.e. $(\frac{1}{2}\pi-\theta) > O(E^{\frac{1}{5}})$, the dominant terms in the stream-function equation are

$$-4\zeta^2\frac{\partial^2\psi}{\partial\eta^2}=\frac{\partial^6\psi}{\partial\eta^6}. \tag{5.5}$$

This is the classical Ekman-layer equation, whose solution is well known. When ζ is small, i.e. $(\frac{1}{2}\pi-\theta) < O(E^{\frac{1}{5}})$, the dominant terms in the stream-function equation are

$$-4(1+N^2\sigma)\frac{\partial^2\psi}{\partial\zeta^2}=\frac{\partial^6\psi}{\partial\eta^6}. \tag{5.6}$$

In the intermediate region where $\frac{1}{2}\pi-\theta = O(E^{\frac{1}{5}})$, all the terms in (5.4) are important. Let the interface of this region and the Ekman layer be given by $\zeta = \zeta_I$, where ζ_I is $O(1)$.

We shall obtain the solution to the Ekman-layer equation (5.5) and evaluate this solution at the interface ζ_I. We assume that this function is an approximate solution to (5.4) in the neighbourhood of ζ_I, and we shall take this function to be the upper boundary condition on (5.6) at $\zeta = \zeta_I$. We then solve (5.6) with such a boundary condition at $\zeta = \zeta_I$, assume zero flow across the equator, and match the interior flow with the surface stress at $\eta = 0$. The composite solution in the regions $\zeta \geqslant \zeta_I$ and $\zeta \leqslant \zeta_I$, which is continuous but not smooth at $\zeta = \zeta_I$, is then an approximate solution for the equatorial boundary layer. We note that this solution is in reasonable agreement with a numerical solution of (5.4) given by Dowden (1972).

We shall now carry out the procedure outlined above in the case where the stress is $\tau = A\sin\theta$. It has already been shown that such a stress generates an interior velocity

$$u_3 = 5Art\sin\theta+\tfrac{1}{2}Ar^3\sin^3\theta. \tag{5.7}$$

Classical Ekman-layer analysis gives the stream-function solution of (5.5) as

$$\psi = \tfrac{1}{2}A\zeta\exp(-\zeta^{\frac{1}{2}}\eta)\cos\zeta^{\frac{1}{2}}\eta. \tag{5.8}$$

Hence, $\psi = \frac{1}{2}A\zeta_I\exp(-\zeta_I^{\frac{1}{2}}\eta)\cos\zeta_I^{\frac{1}{2}}\eta$ at $\zeta = \zeta_I$. The boundary conditions in the equatorial layer at the side wall $\eta = 0$ become

$$\frac{\partial\tilde{u}_3}{\partial\eta} = -A\zeta^2, \quad \frac{\partial\tilde{u}_2}{\partial\eta} = 0, \quad \tilde{u}_1 = \tfrac{1}{2}A \quad \text{at} \quad \eta = 0, \tag{5.9}$$

where a tilde denotes a boundary-layer quantity. As usual in such boundary-layer analysis, the condition as $\eta \to \infty$ is exponential decay. Symmetry prescribes zero flow across the equator, thus

$$\tilde{u}_2 = 0 \quad \text{at} \quad \zeta = 0. \tag{5.10}$$

Hence we seek the solution to (5.6) with boundary conditions given by (5.8)–(5.10), namely

$$-4(1+N^2\sigma)\frac{\partial^2\psi}{\partial\zeta^2} = \frac{\partial^6\psi}{\partial\eta^6},$$

$$\left.\begin{aligned} -\zeta\psi + \int_\infty^0 \frac{\partial\psi}{\partial\eta}\,d\eta &= -\tfrac{1}{2}A\zeta^2 \\ \partial^2\psi/\partial\eta^2 = 0, \quad \psi &= \tfrac{1}{2}A \end{aligned}\right\} \quad \text{at} \quad \eta = 0,$$

$$\psi \to 0 \quad \text{as} \quad \eta \to \infty,$$

$$\psi = 0 \quad \text{at} \quad \zeta = 0,$$

$$\psi = \tfrac{1}{2}A\zeta_I \exp(-\zeta_I^{\frac{1}{2}}\eta)\cos\zeta_I^{\frac{1}{2}}\eta \quad \text{at} \quad \zeta = \zeta_I.$$

This problem has a non-singular solution given by

$$\psi = \psi_1 + \tfrac{1}{2}A\zeta \exp(-\zeta_I^{\frac{1}{2}}\eta)\cos\zeta_I^{\frac{1}{2}}\eta,$$

where

$$\psi_1 = \sum_{n=1}^{\infty} \sin\frac{n\pi\zeta}{\zeta_I}[A_{1n}\exp(\omega_{1n}\eta) + A_{2n}\exp(\omega_{2n}\eta) + A_{3n}\exp(\omega_{3n}\eta) + C_{1n}\exp(-\zeta_I^{\frac{1}{2}}\eta)\sin\zeta_I^{\frac{1}{2}}\eta + C_{2n}\exp(-\zeta_I^{\frac{1}{2}}\eta)\cos\zeta_I^{\frac{1}{2}}],$$

with

$$(\omega_{1n}, \omega_{2n}, \omega_{3n}) = 4(1+N^2\sigma)^{\frac{1}{6}}\frac{n\pi^{\frac{1}{3}}}{\zeta_I}\left(-1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\right),$$

$$C_{2n} = \frac{A(-1)^{n+1}\zeta_I}{n\pi}\left[\frac{(n\pi)^4}{\zeta_I^{10}}\left(\frac{1+N^2\sigma}{2}\right)^2 - 1\right],$$

$$C_{1n} = -\left(\frac{1+N^2\sigma}{2}\right)\frac{(n\pi)^2}{\zeta_I^5}C_{2n},$$

$$A_{1n} = \frac{1}{2}\left[C_{1n}\zeta_I + \frac{C_{1n}+C_{2n}}{2\zeta_I^{\frac{1}{2}}}\right],$$

$$A_{2n} = \frac{3-i\sqrt{3}}{6}\left[-C_{2n} - C_{1n}\zeta_I\left(\frac{1+i\sqrt{3}}{4}\right) - \frac{(C_{1n}+C_{2n})}{2\zeta_I^{\frac{1}{2}}}\left(\frac{3-i\sqrt{3}}{4}\right)\right],$$

$$A_{3n} = \frac{3+i\sqrt{3}}{6}\left[-C_{2n} - C_{1n}\zeta_I\left(\frac{1-i\sqrt{3}}{4}\right) - \frac{(C_{1n}+C_{2n})}{2\zeta_I^{\frac{1}{2}}}\left(\frac{3+i\sqrt{3}}{4}\right)\right],$$

where $\tfrac{1}{2}\pi - \theta_I = E^{\frac{1}{6}}\zeta_I$ and ζ_I is $O(1)$, $0 \leqslant \eta < \infty$, $0 \leqslant \zeta \leqslant \zeta_I$.

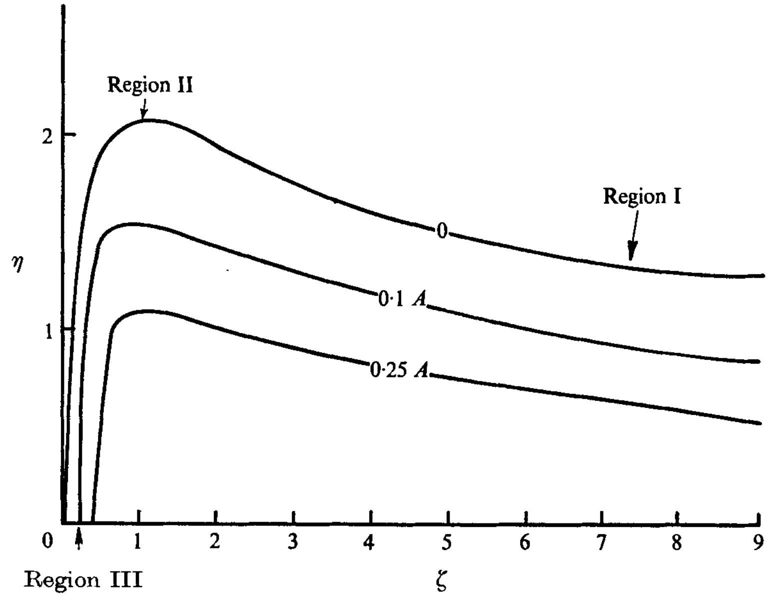

FIGURE 3. Sketch of contours of u_2 velocity component in the boundary layer. Region I, $\frac{1}{2}\pi-\theta > O(E^{\frac{1}{5}})$. Region II, $\frac{1}{2}\pi-\theta = O(E^{\frac{1}{5}})$. Region III, $\frac{1}{2}\pi-\theta < (E^{\frac{1}{5}})$.

From the expression for the stream function, we can calculate the southward component $\tilde{u}_2$ of the velocity:

$$\tilde{u}_2 = \tfrac{1}{2}A\zeta\zeta_I^{\frac{1}{2}}\exp\left(-\zeta_I^{\frac{1}{2}}\eta\right)\left(\cos\zeta_I^{\frac{1}{2}}\eta+\sin\zeta_I^{\frac{1}{2}}\eta\right)$$
$$-\sum_{n=1}\sin\frac{n\pi}{\zeta_I}\zeta[A_{1n}\omega_{1n}\exp(\omega_{1n}\eta)+A_{2n}\omega_{2n}\exp(\omega_{2n}\eta)+A_{3n}\omega_{3n}\exp(\omega_{3n}\eta)$$
$$+C_{1n}\zeta_I^{\frac{1}{2}}\exp\left(-\zeta_I^{\frac{1}{2}}\eta\right)\left(\cos\zeta_I^{\frac{1}{2}}\eta-\sin\zeta_I^{\frac{1}{2}}\eta\right)C_{2n}\zeta_I^{\frac{1}{2}}\exp\left(-\zeta_I^{\frac{1}{2}}\eta\right)\left(\cos\zeta_I^{\frac{1}{2}}\eta+\sin\zeta_I^{\frac{1}{2}}\eta\right)].$$

In the Ekman layer, the $\tilde{u}_2$ velocity component is

$$\tilde{u}_2 = \tfrac{1}{2}A\zeta^{\frac{3}{2}}\exp\left(-\zeta^{\frac{1}{2}}\eta\right)\left(\cos\zeta^{\frac{1}{2}}\eta+\sin\zeta^{\frac{1}{2}}\eta\right).$$

Figure 3 is a sketch of the level curves of u_2 in the regions I, II and III of the boundary layer.

We note that the analytic solution does not vary smoothly between region I and region II. This is to be expected from our method of attacking the problem, since we have only approximated the solution in region II by the value at ζ_I. In figure 3, however, the discontinuity has been smoothed to give a sketch of the realistic solution approximated by our analysis. To obtain a smooth curve rigorously, it is necessary to solve the full equation (5.7). This has been done numerically by Dowden (1972). His results give smooth curves of the general shape we have obtained; he also concludes that the u_2 velocity component at the surface increases from zero to a maximum which falls away as the solution becomes more like an Ekman layer. On rescaling the Dowden co-ordinates to be equivalent to ζ and η, we see that the numerical value of ζ_I given by his results is approximately unity.

We conclude that our method of investigation of the equatorial boundary layer gives a good approximation to the behaviour of ψ and u_2. We have obtained a non-singular solution which agrees with the Ekman-layer solution away from the equator and which matches the interior flow with the prescribed boundary conditions close to the equator. Hence, in the circumstances we are considering, the role of the equatorial layer is essentially passive. The interior flow is driven by the classical Ekman-layer circulation and is not modified to first order by equatorial effects.

6. Comparison with numerical results

In the previous sections we have exhibited the flow obtained on a long time scale for stress-driven flow in a rotating stratified sphere. We have obtained explicit expressions describing the pressure field and have shown that the role, in this problem, of the equatorial boundary layer is essentially a passive one. The quasi-steady flow is achieved on a time scale $O(N^2\sigma E^{-1})$: before this time is reached, the flow is in a transitional stage where the circulation is driven by a combination of Ekman-layer suction and thermal diffusion.

In obtaining the above results, we made use of several simplifying approximations. In support of our methods for attacking the problem, we now observe that our results are in good agreement with a numerical model developed by Moore & Weir (1976) for spin-up in a rotating stratified sphere. The numerical model gives a stream function for flow on a long time scale that is in very good qualitative agreement with the cellular pattern (see figure 2) obtained from our analytic treatment of the problem. Their numerical work also shows that in our parameter range there is a very small distortion of the temperature field. Comparison between the two sets of results describing the boundary-layer structure is also good: the numerical model shows that the Ekman layer is unaffected by stratification. We also see that, while the boundary layer thickens at the equator, there is no significant jet of fluid affecting the interior in equatorial regions. We finally observe that the numerical model is based on the full nonlinear system of equations. The results, however, indicate that the effects of nonlinearity are very small, thus supporting our use of the linear approximation.

7. Relevance to the solar problem

We have obtained the above results by considering parameters relevant to the solar problem, where $N^2\sigma \ll 1$. We were particularly interested in this case in light of the controversy surrounding hypotheses of Dicke (1964, 1967) concerning the nature of the sun. Is it possible that the strong density gradient in the core and the large-scale stellar dimensions would make a solar Ekman layer unimportant and allow differential rotation between the interior core and the outer shell to persist over the lifetime of the sun? We model the effect of the viscous coupling between the more slowly rotating outer shell and the inner core by considering a steady stress applied to the surface of a rotating stratified sphere. Our results show that on a time scale $O(N^2\sigma E^{-1})$ the stress is completely communicated to the interior; we obtain a quasi-steady flow that is governed by a combination of Ekman-layer driving and thermal diffusion. It is true as Dicke suggests that strong stratification inhibits the role of the Ekman layer, but the circulation is not entirely blocked. Considering parameters relevant to the solar problem, our results show that an initial differential rotation between the core and the outer shell would be smoothed out on a time scale of order 10^9 years. It should be noted that this is very close to the lifetime of the sun, and our analysis indicates that on a shorter time scale the flow is still transient. On this shorter time scale, the effects of the stress have not fully penetrated the interior, and a central portion of the core could be rotating more rapidly. However, as Howard, Moore & Spiegel (1967) remark, it is more realistic to consider the boundary layers as turbulent, which considerably shortens the spin-down time. Thus 10^9 years is an upper bound for the achievement of quasi-steady circulation of the solar core.

We should also remark that our analysis has made use of the fact that both E and $N^2\sigma$ are small parameters. The double expansion assumes that

$$E^{\frac{1}{2}} \ll N^2\sigma \ll 1.$$

This is indeed the case in the solar problem, where $E^{\frac{1}{2}} \sim 10^{-8}$ and $N^2\sigma \sim 10^{-4}$. If we were to continue further in the expansion in powers of $N^2\sigma$ we should have to consistently include second-order terms from the original $E^{\frac{1}{2}}$ expansion. However our double expansion is valid to $O(N^2\sigma)$.

As we mentioned in the introduction, the problem of flow in a rotating stratified fluid has been previously investigated by Friedlander (1974) and Sakurai *et al.* (1971) for parameter ranges relevant to the solar problem. They consider flow of a fluid with small Prandtl number in a cylindrical geometry. The approach of Friedlander (1974) is similar to the approach taken in the current investigation of the problem in a sphere: the results and conclusions are the same for both geometries. In the work of Sakurai *et al.* (1971), motion driven by an imposed velocity on the boundary, $u_2 = \omega r$, is considered. With this boundary condition, they also conclude that the asymptotic state of rigid rotation is achieved on a time scale of order $N^2\sigma E^{-1}$. In order to represent the interaction of the shell of the sun with the solar core, they then consider a sequence of impulses with decreasing imposed angular velocity. The process by which this modified boundary condition is communicated to the interior is less efficient and it is possible that the angular velocity in the solar core is non-uniform and the impulses have not fully penetrated the interior in the lifetime of the sun.

In our present analysis of the problem, we model the effect of the more slowly rotating solar convection zone on the interior core by considering a steady surface stress. We conclude that on a time scale of order $N^2\sigma E^{-1}$ the boundary driving force has fully penetrated the interior and the fluid motion is in a quasi-steady state where the velocity is given by (4.1). The long term state of rigid rotation growing linearly with time is not however achieved until the viscous time scale $O(E^{-1})$ has elapsed. The difference between the driving mechanisms of Friedlander and Sakurai *et al.* appears to lead to certain differences in the conclusions. Both conclude that rigid rotation will not be achieved until a time has elapsed that is longer than the lifetime of the sun. However, in the stress-driven model, the fluid is spun down to an intermediate steady state on a time scale that is within the lifetime of the sun.

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